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The Adventitious Angles Problem: The Lonely Fractional Derived Angle

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Abstract. In the "classical" adventitious angle problem, for a given set of three angles a, b, and c measured in integral degrees in an isosceles triangle, a fourth angle θ (the derived angle), also measured in integral degrees, is sought. We generalize the problem to find θ in fractional degrees. We show that the triplet $(a, b, c) = (45^\circ, 45^\circ, 15^\circ)$ is the only combination that leads to $\theta = 7\frac{1}{2}^\circ$ as the fractional derived angle.

The familiar problem of adventitious angles deals with an isosceles triangle as illustrated in Figure. 1 where $AB = AC$. The problem asks to find the angle θ (the *derived* angle) for certain given values of the angles a, b, and c. In the following we will use the notation $(a, b, c; \theta)$ to denote a problem and its solution.

The problem is also known as Langley's problem, who first proposed the particular case of $(20^\circ, 60^\circ, 50^\circ; 30^\circ)$ [**2**]. The angles are called "adventitious" due to the fact that among 113564 possible triplets of a, b, and c (with disregard for mirror images and trivial cases of $b = c$), only 53 of them yield an angle of θ with integral numbers of degrees [**7**].

These seemingly innocent problems are more difficult than their simple appearances [**1**]. Straightforward use of angle chasing will not be sufficient to solve these problems. Ingenious constructions are usually needed [**4, 5, 6, 7**].

In the "classical" adventitious angle problem, the given angles a, b, and c as well as the derived angle θ are all measured in integral degrees. We generalize the problem to find θ in fractional degrees while still keeping the angles a, b, and c in integral degrees. We'll first prove that for the angle θ to be fractional degrees, its denominator is at most 2. A computer search shows that $(45^\circ, 45^\circ, 15^\circ; 7\frac{1}{2}^\circ)$ is the only potential fractional solution. For this particular case we'll give a pure geometrical proof and a trigonometric proof that $(45^\circ, 45^\circ, 15^\circ)$ is the only adventitious set of angles that has fractional derived angle $\theta = 7\frac{1}{2}^\circ$.

Theorem 1. *For the angle θ to be a rational number of degrees when angles a, b, and c are measured in integral degrees, the denominator of θ is at most 2.*

To prove the theorem we first prove the following three lemmas. In the following, $\mathbb{Q}$ and $\mathbb{R}$ stand for the fields of rational and real numbers. Let n be a positive integer and $\alpha = \frac{2\pi}{n}$. Let $i = \sqrt{-1}$ and denote ζ_n an nth root of unity $e^{\frac{2\pi i}{n}} = e^{\alpha i}$, $\mathbb{Q}_n = \mathbb{Q}(\zeta_n)$ the nth cyclotomic field, and $\mathbb{Q}_n^+$ the maximal real subfield of $\mathbb{Q}_n$.

Lemma 1. *If* $4|n$, *then* $\mathbb{Q}_n^+ = \mathbb{Q}_n \cap \mathbb{R} = \mathbb{Q}\left(\tan(\frac{\alpha}{2})\right)$.

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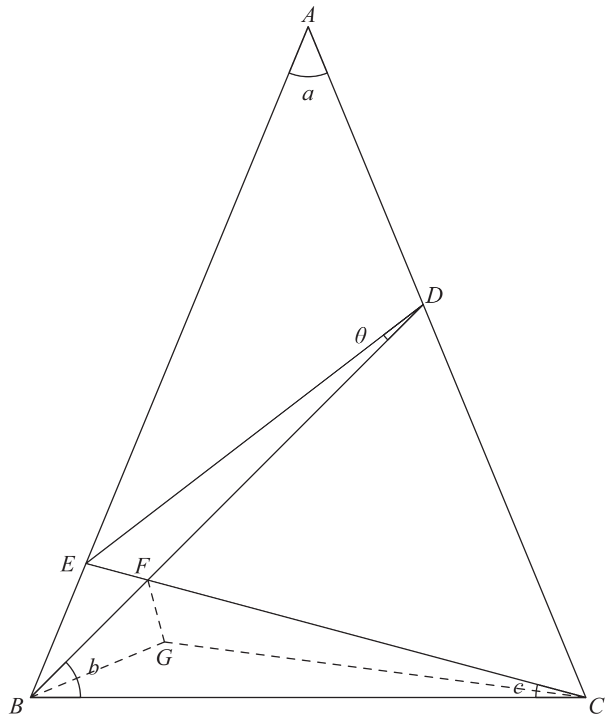

Figure 1.

Proof. By definition $\mathbb{Q}_n^+ = \mathbb{Q}\left(\zeta_n + \zeta_n^{-1}\right) = \mathbb{Q}(\cos\alpha)$. Since $4|n$, we have $i \in \mathbb{Q}_n$. Hence $\sin\alpha = (\zeta_n - \zeta_n^{-1})/(2i) \in \mathbb{Q}_n^+$. The identity $\tan(\frac{\alpha}{2}) = \frac{1-\cos\alpha}{\sin\alpha}$ shows $\tan(\frac{\alpha}{2}) \in \mathbb{Q}_n^+$, so $\mathbb{Q}\left(\tan(\frac{\alpha}{2})\right) \subset \mathbb{Q}_n^+$. On the other hand, $\cos\alpha = \frac{1-\tan^2(\frac{\alpha}{2})}{1+\tan^2(\frac{\alpha}{2})}$, so $\mathbb{Q}_n^+ = \mathbb{Q}(\cos\alpha) \subset \mathbb{Q}\left(\tan(\frac{\alpha}{2})\right)$. ■

Lemma 2. *Assume* $4|n$ *and* $4|m$. *If* $\mathbb{Q}_m^+ \subset \mathbb{Q}_n^+$, *then* $m|n$.

Proof. From the assumptions we have $\mathbb{Q}_m = \mathbb{Q}_m^+(i) \subset \mathbb{Q}_n^+(i) = \mathbb{Q}_n$. When n is even, the only roots of unity in $\mathbb{Q}_n$ are the nth roots of unity [**3**, Corollary 3, p. 19]. Since $\zeta_m \in \mathbb{Q}_m \subset \mathbb{Q}_n$, ζ_m must be one of the nth roots of unity: $\zeta_m^n = e^{\frac{2\pi i n}{m}} = 1$. This happens if and only if $m|n$. ■

Lemma 3. *If* $\beta = \frac{2\pi}{360m}$ *for some integer m such that* $\tan\beta \in \mathbb{Q}_{360}^+$, *then* $m|2$.

Proof. From Lemma 1, $\mathbb{Q}_{180m}^+ = \mathbb{Q}(\tan\beta) \subset \mathbb{Q}_{360}^+$, hence $180m|360$ by Lemma 2, so $m|2$. ■

Proof. Theorem 1.

A formula for $\tan\theta$ can be obtained by applications of sine rule [**7**]

$$\tan\theta = \frac{\sin(b+c)\sin(c)(\cos a + \cos 2b)}{\sin(b)(\cos a + \cos 2c) + \cos(b+c)\sin(c)(\cos a + \cos 2b)}.$$

The conclusion follows from Lemma 3. ■

Theorem 2. *The only fractional derived angle, when angles a, b, and c are integral numbers of degrees, is $\theta = 7\frac{1}{2}^\circ$ when the triplet $(a, b, c) = (45^\circ, 45^\circ, 15^\circ)$.*

Proof. A quick computer search shows that $(45^\circ, 45^\circ, 15^\circ; 7\frac{1}{2}^\circ)$ is the only potential case. The search is carried out using software package Maple (version 12) with 100 decimal digits. We will give two proofs that $(45^\circ, 45^\circ, 15^\circ; 7\frac{1}{2}^\circ)$ is a solution: an elementary geometry proof and a trigonometric proof.

For the geometrical proof, let G be the incenter of triangle $\triangle BFC$ where three angle bisectors of the triangle meet (Figure 1). Since $\angle EBF = \angle GBF = 22\frac{1}{2}^\circ$ and $\angle BFE = \angle BFG = 60^\circ$, it follows that $\triangle BEF \cong \triangle BGF$. Hence $BE = BG$. Since $\angle BDC = \angle DCB = 67\frac{1}{2}^\circ$, we have $BC = BD$, which leads to $\triangle BED \cong \triangle BGC$. Hence $\theta = \angle EDB = \angle GCB = 7\frac{1}{2}^\circ$.

For the trigonometric proof, we use the identity from Eq. (6) of [**4**]:

$$\frac{\sin\theta}{\sin(b+c-\theta)} = \frac{\cos(b+\frac{a}{2})\sin(c)\cos(b-\frac{a}{2})}{\cos(c-\frac{a}{2})\sin(b)\cos(c+\frac{a}{2})}.$$

Substituting the values of a, b, and c into both sides of this identity, and simplifying the right side, we have

$$\begin{aligned}\frac{\sin\theta}{\sin(\frac{\pi}{4}+\frac{\pi}{12}-\theta)} &= \frac{\cos\frac{3\pi}{8}\sin\frac{\pi}{12}\cos\frac{\pi}{8}}{\cos\frac{\pi}{24}\sin\frac{\pi}{4}\cos\frac{5\pi}{24}} \\ &= \frac{(2\sin\frac{\pi}{8}\cos\frac{\pi}{8})\sin\frac{\pi}{24}\sin\frac{\pi}{12}}{(2\sin\frac{\pi}{24}\cos\frac{\pi}{24})\sin\frac{\pi}{4}\cos\frac{5\pi}{24}} \\ &= \frac{\sin\frac{\pi}{24}}{\sin(\frac{\pi}{4}+\frac{\pi}{12}-\frac{\pi}{24})}.\end{aligned}$$

Hence $\theta = \frac{\pi}{24} = 7\frac{1}{2}^\circ$. ∎

Theorem 1 can be generalized to basic units other than degree $(\pi/180)$ [**7**].

Corollary 1. *Given the set of angles a, b, and c each in multiples of π/N radians where N is a positive integer divisible by 4, if $\theta = k\pi/N$ and k is a rational number, then the denominator of k is at most 2.*

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